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LETTER TO THE EDITOR

Functional integral representation of operators and the Weyl quantization rule

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Abstract. The Weyl quantization rule is shown to follow from a semiclassical expansion of an exact expression for a functional of an operator given by Feynman.

Some years ago Weyl (1950) gave a prescription for associating a classical function $F(p,q)$ of coordinate q and momentum p with the corresponding quantum mechanical function $\hat{F}(\hat{p},\hat{q})$ of operators $\hat{q}$ and $\hat{p}$. An extensive discussion of such rules of association with references to earlier work has been given recently by Agarwal and Wolf (1970). A defect of all of these is that they are all, to an unknown degree, approximations, so that different rules can associate different classical functions $F(p,q)$ with the same operator function $\hat{F}(\hat{\boldsymbol{p}},\hat{\boldsymbol{q}})$.

The purpose of the present letter is to point out that a formalism due to Feynman (1951) enables a unique classical *functional* to be associated with an operator function. An expansion of the WKB type then leads to the Weyl quantization rule as the first term of a series in powers of $\hbar$. The Weyl rule is thus found to have a basis as the first approximation to an exact expression, and corrections to it can, at least in principle, be calculated.

In order to apply the Feynman method we first write for any functional $F[M(s)]$

$$F[M(s)] = \int \exp\left(\mathrm{i}\int_0^1 \mu(s)M(s)\,\mathrm{d}s\right)\mathscr{F}[\mu(s)]\mathscr{D}\mu(s) \tag{1}$$

$$\mathscr{F}[\mu(s)] = \int \exp\left(-\mathrm{i}\int_0^1 \mu(s)M(s)\,\mathrm{d}s\right)F[M(s)]\mathscr{D}M(s) \tag{2}$$

where $\mathscr{F}[\mu(s)]$ is, in general, a complex functional. The operator function $\hat{F}[\hat{\boldsymbol{M}}(s)]$ is then defined by

$$\hat{F}[\hat{\boldsymbol{M}}(s)] = \int \exp\left(\mathrm{i}\int_0^1 \mu(s)\hat{\boldsymbol{M}}(s)\,\mathrm{d}s\right)\mathscr{F}[\mu(s)]\mathscr{D}\mu(s).$$

Noncommuting operators $\hat{\boldsymbol{A}}\hat{\boldsymbol{B}}\hat{\boldsymbol{C}}\ldots$ which occur in a particular order are given labels $\hat{\boldsymbol{A}}(s_1)\,\hat{\boldsymbol{B}}(s_2)\,\hat{\boldsymbol{C}}(s_3)\ldots$ where the convention is adopted that 'times' increase going from right to left, so that $s_1 > s_2 > s_3\ldots.$ The 'time' dependence of $\hat{M}$ is thus a consequence of the order of the original operators. An arbitrary operator functional $\hat{F}[\hat{p}(s),\hat{q}(s)]$ may be similarly written

$$\hat{F}[\hat{\boldsymbol{p}}(s),\hat{\boldsymbol{q}}(s)] = \iint \exp \mathrm{i}\int_0^1 \mu(s)\hat{\boldsymbol{p}}(s)\,\mathrm{d}s \exp \mathrm{i}\int_0^1 \tau(s)\hat{\boldsymbol{q}}(s)\,\mathrm{d}s\mathscr{F}[\mu(s),\tau(s)]\mathscr{D}\mu(s)\mathscr{D}\tau(s) \tag{3}$$

where the ordering of the exponential functions is immaterial since the functional $\mathscr{F}[\mu(s),\tau(s)]$ puts the operators in their proper order. A classical functional $F[(p(s),q(s))]$ can be associated with (3). We now make use of the following theorem proved by Feynman (1951):

$$\exp\left(\int_0^1 \boldsymbol{P}(s)\,\mathrm{d}s\right) F[\hat{\boldsymbol{M}}(s),\hat{\boldsymbol{N}}(s)\ldots] = \boldsymbol{U}(1)F[\hat{\boldsymbol{M}}'(s),\hat{\boldsymbol{N}}'(s)\ldots]\boldsymbol{U}(0)$$

where

$$\boldsymbol{M}'(s) = \boldsymbol{U}^{-1}(s)\boldsymbol{M}(s)\boldsymbol{U}(s) \qquad \text{and} \qquad \boldsymbol{U}(s) = \exp\int_0^s \boldsymbol{P}(s')\,\mathrm{d}s'. \tag{4}$$

Applying this to (3) we obtain

$$\begin{aligned}\hat{F}[\hat{\boldsymbol{p}}(s),\hat{\boldsymbol{q}}(s)] &= \iint \exp \mathrm{i}\hat{\boldsymbol{p}}(1)\int_0^1 \mu(s)\,\mathrm{d}s \exp \mathrm{i}\int_0^1 \tau(s)\left(\hat{\boldsymbol{q}}(0)+\hbar\int_0^s \mu(s)\,\mathrm{d}s\right)\\ &\quad\times \mathscr{F}[\mu(s),\tau(s)]\,\mathrm{d}\mu(s)\,\mathrm{d}\tau(s)\\ &= \iint \exp \mathrm{i}\left\{\hat{\boldsymbol{p}}(1)\int_0^1 \mu(s)\,\mathrm{d}s + \int_0^1 \tau(s)\left(\hat{\boldsymbol{q}}(0)+\hbar\int_0^s \mu(s')\,\mathrm{d}s'\right)\right\}\mathrm{d}s\\ &\quad\times \mathscr{F}[\mu(s),\tau(s)]\,\mathrm{d}\mu(s)\,\mathrm{d}\tau(s)\end{aligned} \tag{5}$$

where we can replace $\hat{\boldsymbol{q}}(s)$ by $\hat{\boldsymbol{q}}(0)$ since all these precede $\hat{\boldsymbol{p}}(1)$. As a first approximation we can put $\hbar = 0$ in the second exponential. Regarding $\int_0^1 \mu(s)\mathrm{d}s$, $\int_0^1 \tau(s)\,\mathrm{d}s$ as Riemann sums we define the $2N$ new variables

$$\begin{aligned}\alpha_0 &= \sum_i \mu(s_i)\Delta, \qquad \alpha_1 = \mu(s_1)-\mu(s_2), \qquad \ldots, \qquad \alpha_{n-1} = \mu(s_{n-1})-\mu(1)\\ \beta_0 &= \sum_i \tau(s_i)\Delta, \qquad \beta_1 = \tau(s_1)-\tau(s_2), \qquad \ldots, \qquad \beta_{n-1} = \tau(s_{n-1})-\tau(1).\end{aligned} \tag{6}$$

The integration over all the variables except α_0, β_0 can be carried out and gives a function $G(\alpha_0,\beta_0)$. This leaves

$$\hat{G}[\hat{\boldsymbol{p}}(1),\hat{\boldsymbol{q}}(0)] = \iint \exp\mathrm{i}[\hat{\boldsymbol{p}}(1)\alpha_0+\hat{\boldsymbol{q}}(0)\beta_0]G[\alpha_0,\beta_0]\,\mathrm{d}\alpha_0\,\mathrm{d}\beta_0 \tag{7}$$

with a corresponding expression for the classical function $G(p(1), q(0))$. Equation (7) is just the Weyl quantization rule, which has thus been derived as a semiclassical approximation to the exact expression (3). It may be noted that this derivation is very similar in spirit to the semiclassical expansion of the quantum mechanical partition function when this is represented as a path integral (Gelfand and Yaglom 1960).

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